REGULARIZED INNER PRODUCTS OF MODULAR FUNCTIONS

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Dedicated to the memory of Marvin Knopp

Abstract. In this note we give an explicit basis for the harmonic weak forms of weight two. We also show that their holomorphic coefficients can be given in terms of regularized inner products of weight zero weakly holomorphic forms.

1. Introduction

One of the most basic arithmetic functions is the sum of divisors function

$$\sigma(m) = \sum_{d|m} d$$

defined for m a positive integer. Ramanujan [16] gave a surprising expansion for $\sigma(m)$ as an infinite sum:

$$\sigma(m) = \frac{\pi^2 m}{6}\left(1 + \frac{(-1)^m}{2^2} + \frac{2\cos\frac{2}{3}m\pi}{3^2} + \frac{2\cos\frac{1}{4}m\pi}{4^2} + \frac{2(\cos\frac{2}{5}m\pi + \cos\frac{4}{5}m\pi)}{5^2} + \cdots\right). \tag{1.1}$$

The numerator over c^2 is the Ramanujan sum $\sum_{(a,c)=1}\cos(\frac{2\pi m a}{c})$. This identity clearly displays the oscillations of $\sigma(m)$ around its mean value $\frac{\pi^2 m}{6}$. Also, (1.1) makes sense as a limit when $m = 0$ and gives the extension $\sigma(0) = -\frac{1}{24}$.

There is a nice generalization of Ramanujan's formula, that goes back to Petersson and Rademacher, that connects it with the theory of modular forms (see Knopp's beautiful exposition [12]). Consider the sum for $m, n > 0$

$$c_m(n) = 2\pi\sqrt{\tfrac{m}{n}}\sum_{c>0}\frac{K(-m,n,c)}{c} I_1\Big(\frac{4\pi\sqrt{mn}}{c}\Big) \tag{1.2}$$

where $K(m,n,c) = \sum_{(a,c)=1} e(\frac{ma+n\bar{a}}{c})$ is the Kloosterman sum and $I_1(x)$ is the I-Bessel function. By any non-trivial estimate for $|K(m,n,c)|$, (1.2) converges absolutely, using also that $I_1(x) \sim \frac{x}{2}$ as $x \to 0$. The sum (1.2) generalizes (1.1); it makes sense when $n = 0$ and reduces to

$$c_m(0) = 24\sigma(m)$$

after applying (1.1). Also $c_0(n) = 0$ for $n > 0$. Amazingly, the $c_m(n)$ are always integers and their generating function in n given by

$$f_m(z) = q^{-m} + \sum_{n\geq 0} c_m(n) q^n$$

is a modular function for the full modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ when we let, as usual, $q = e^{2\pi i z}$. This well-known fact follows from the general formulas given below.

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In particular, $f_0(z) = 0$ and $f_1(z) = j(z) - 720$, where $j(z) = q^{-1} + 744 + 198664q + \dots$ is the usual modular j-function. More generally,

$$f_m(z) = j_m(z) + 24\sigma(m) \tag{1.3}$$

where $j_m \in \mathbb{C}[j]$ is uniquely determined by having a Fourier expansion of the form

$$j_m(z) = q^{-m} + \sum_{n>0} b_m(n)q^n. \tag{1.4}$$

The f_m for $m > 0$ form a basis for the subspace of $\mathbb{C}[j]$ consisting of modular functions that are orthogonal to the constant function with respect to the regularized Petersson inner product. Let $\mathcal{F}_Y$ be the usual fundamental domain for Γ truncated at $y = Y$ and set for f, g modular functions

$$\langle f, g\rangle_{\mathrm{reg}} = \lim_{Y\to\infty} \int_{\mathcal{F}_Y} f(z)\overline{g(z)}\tfrac{dxdy}{y^2},$$

provided this limit exists. We usually simply write $\langle f, g\rangle = \langle f, g\rangle_{\mathrm{reg}}$. Then we have that for all $m \geq 0$

$$\langle f_m, 1\rangle = 0, \tag{1.5}$$

which follows easily from results of [1]. In this paper we are interested in the quantities $\langle f_m, f_n\rangle$ when $m \neq n$, which are finite. An interesting question that we have not been able to answer concerns their possible arithmetic or geometric meaning and in particular the case of $m = n$.

Using Hecke operators it is enough to consider the sequence $\langle f_m, f_1\rangle$ for $m > 1$. Here are some of its values computed numerically:

$$\langle f_2, f_1\rangle = 366.765,\ \langle f_3, f_1\rangle = 195.677,\ \langle f_4, f_1\rangle = 501.665$$

As an application of our main result, we will give a formula of the type (1.2) for $\langle f_m, f_n\rangle$.

Theorem 1. *For unequal positive integers m, n we have*

$$\langle f_m, f_n\rangle = -8\pi^2\sqrt{mn}\sum_c \frac{K(m,n,c)}{c}F\Big(\frac{4\pi\sqrt{mn}}{c}\Big), \tag{1.6}$$

where $F(u) = \pi Y_1(u) + \frac{2}{u}J_0(u)$, where J_s is the Bessel function of the first kind, and Y_s is the Bessel function of the second kind.

Since the Bessel function of second kind $Y_1(u)$ satisfies

$$\frac{\pi}{2}Y_1(u) = -\frac{1}{u} + J_1(u)\log(u/2) + O(u)$$

as $u \to 0$ we have that $F(u) = O(u \log u)$ as $u \to 0$ and the above series converges.

The proof of (1.5) uses the fact that the generating function of $\sigma(m)$ given by

$$E_2^*(z) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n - \frac{3}{\pi y}, \tag{1.7}$$

is a harmonic Eisenstein series of weight 2. Here and throughout the paper $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. Our proof of Theorem 1 makes use of harmonic weak Maass forms of weight 2 constructed using Poincaré series. To explain how we must introduce some notation. Let $\mathcal{H}$ denote the upper half plane. Recall that for any $k \in \mathbb{R}$, the Maass-type differential operator ξ_k is defined through its action on a differentiable function f on $\mathcal{H}$ by

$$\xi_k(f) = 2iy^k\overline{\tfrac{\partial f}{\partial \bar{z}}}.$$

It is easily checked that

$$\xi_k\big((\gamma z+\delta)^{-k} f(gz)\big) = (\gamma z+\delta)^{k-2}(\xi_k f)(gz)$$

for any $g = \pm\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{R})$. Thus if $f(z)$ has weight k for Γ then $\xi_k f$ has weight $2-k$ and $\xi_k f = 0$ if and only if f is holomorphic. The weight k Laplacian can be conveniently defined by

$$\Delta_k = -\xi_{2-k} \circ \xi_k \tag{1.8}$$

If f is a real analytic function on $\mathcal{H}$ of weight k for Γ that is harmonic on $\mathcal{H}$ in the sense that

$$\Delta_k f = 0$$

then f will have a Fourier expansion at $i\infty$ each of whose terms has at most linearly exponential growth. Such an f is called a *harmonic weak Maass form* if it has only finitely many such growing terms. The space of all such forms is denoted by $H_k^!$. It is clear that the space of weakly holomorphic modular forms $M_k^!$, the forms that are holomorphic on $\mathcal{H}$ and meromorphic at infinity, is a subset of $H_k^!$. It follows easily from its general properties that ξ_k maps $H_k^!$ to $M_{2-k}^!$ with kernel $M_k^!$. We note that the space $H_k^!$ is same as $H_k(\mathrm{SL}(2,\mathbb{Z}))$ defined in [4]. Since in almost all the subsequent papers following [4] the notation H_k is used to denote the subspace H_k^+ of harmonic weak Maass forms which is defined as the preimage of cusp forms of weight $2-k$ under ξ_k, we use the notation $H_k^!$. It is easy to see that for $2 < k \in \mathbb{Z}$ the space of harmonic weak Maass forms whose Fourier expansions have no exponentially growing terms is equal to M_k, whereas for $k=2$ it is 1-dimensional and spanned by the non-holomorphic, harmonic Eisenstein series E_2^*.

Our next aim is to construct an explicit basis $\{h_m(z)\}_{m\in\mathbb{Z}}$ for $H_2^!$. It is known that a basis for the weakly holomorphic modular forms of weight 2 is given by $\{f_m'(z)\}_{m>0}$ where we set $f_m'(z) = \frac{-1}{2\pi i}\frac{d}{dz} f_m(z)$ and f_m was defined in (1.3). The basis $\{h_m\}$ constructed in this paper completes the basis $\{f_m'\}$ of $M_2^!$ to a basis $H_2^!$. Namely for $m<0$, $h_m(z) = f_{|m|}'(z)$ whereas for $m>0$ the functions h_m satisfy $\xi_2(h_m) = \frac{1}{4\pi} f_m(z)$. The existence of such harmonic forms was already proved in [4]. Our aim here is to construct this basis explicitly which allows us to prove the Ramanujan-Rademacher type formula for the inner products $< f_m, f_n >$.

In the case of weight $k = 1/2$ such a basis was constructed in [6] and shown there that their holomorphic coefficients can be given in terms of periods of weight zero weakly holomorphic forms along closed geodesics. In [7], on the other hand these coefficients are shown to be equal to regularized inner products of weakly holomorphic forms of dual weight 3/2. Here we show that a similar result also holds for weight $k=2$. Theorem 1 follows from the next result and an explicit formula for the Fourier coefficients given later.

Theorem 2. *For each $m \in \mathbb{Z}$ there is a unique $h_m \in H_2^!$ with Fourier expansion of the form*

$$h_m(z) = \mathcal{M}_m(y)e(mx) + \sum_{n\in\mathbb{Z}} a_m(n)\mathcal{W}_n(y)e(nx) \tag{1.9}$$

where the special function $\mathcal{W}_n(y)$ and $\mathcal{M}_n(y)$ are given in (2.4) and (2.5) in terms of two linearly independent solutions the classical Whittaker differential equation. The function $\mathcal{W}_n(y)$ decays whereas $\mathcal{M}_n(y)$ grows exponentially.

The set $\{h_m\}_{m\in\mathbb{Z}}$ forms a basis for $H_2^!$. We have that $h_0(z) = E_2^*(z)$ and for $m<0$, $h_m(z) = f_{|m|}'(z)$, while for $m>0$ we have

$$\xi_2(h_m(z)) = \tfrac{1}{4\pi} f_m(z). \tag{1.10}$$

The coefficients $a_m(n)$ satisfy the symmetry relation

$$a_m(n) = a_n(m) \tag{1.11}$$

for all integers m, n. Moreover for $m > 0$ we have

$$a_m(n) = \begin{cases} |n| c_m(|n|), & \textit{if } n < 0; \\ -\frac{1}{4\pi} < f_m, f_n > & \textit{if } n > 0,\ m \neq n. \end{cases} \tag{1.12}$$

The Poincaré series formed by averaging the classical Whittaker functions provide a standard tool for constructing harmonic weak Maass forms. There are two subtle points in this construction. First of all when the weight is small, as in the case of $k = 2$ of this paper, such Poincaré series have to be analytically continued. Secondly, at the "harmonic point" $s = k/2$ the Whittaker functions $M_{k/2,s-1/2}$ and $W_{k/2,s-1/2}$ coalesce into the exponential function, and the Poincaré series lead only to holomorphic or weakly holomorphic modular forms. A second solution of the Whittaker differential equation can be constructed using the derivative in one of the parameters of the Whittaker functions. We average such solutions and take their differences to construct a Poincaré series that is harmonic and not weakly holomorphic.

The methods of this paper easily generalize to give a basis $\{h_{m,k}\}$ for $H_k^!$ of harmonic weak Maass forms for weights $k > 2$. In the general case the functions $h_{m,k}$ satisfy $\xi_k(h_m) = f_{m,2-k}$ where $f_{m,2-k} \in M_{2-k}^!$ are the basis functions of weakly holomorphic modular forms of weight $2 - k$ constructed in [8]. Their holomorphic coefficients can also be written in terms of the regularized inner product $< f_{m,2-k}, f_{n,2-k} >$. In the higher weight case there is no issue of analytic continuation, hence from this analytic point of view it is easier. On the other hand for higher weights the space of cusp forms is not trivial and one has include the holomorphic exponential Poincaré series into the basis. Similar constructions of harmonic weak forms using derivatives appeared recently for weight 3/2 in [11]. Also in [2] Bruggeman gives an existence theorem generalizing results of [4] to arbitrary complex weights.

The rest of the paper is organized as follows. For parts of the paper we do not restrict to the case $k = 2$ and work with general weights. In Section 2 we collect information about the Whittaker functions and harmonic forms. In section 3 we write down Poincare series and give their analytic continuation. In Section 4 we construct harmonic forms that are not weakly holomorphic and construct the basis $\{h_m\}$ of Theorem 2. Finally in the last section we prove the inner product formula for the holomorphic coefficients of h_m.

2. Harmonic forms and the Whittaker equation

In this section we review the basic definitions and properties of harmonic weak Maass forms and Whittaker functions.

Definition 3 (Definition of harmonic weak Maass forms)**.** *Let $k \in \mathbb{Z}$. A smooth function $f : \mathcal{H} \to \mathbb{C}$ is called a harmonic weak Maass form of weight k if*

(1) $\Delta_k f = 0$.
(2) $f|_k \gamma = f$ *for all* $\gamma \in \Gamma$.
(3) $f(z) = O(e^{ay})$ *as* $y \to \infty$ *for some* $a \in \mathbb{R}$

The space of all such forms is denoted by $H_k^!$.

The Whittaker functions $M_{\kappa,\nu}(y)$, and $W_{\kappa,\nu}(y)$ (see (2.6) and (2.7)) are the standard solutions of Whittaker's differential equation

$$D_{\kappa,\nu} w = w'' + \left(-\frac{1}{4} + \frac{\kappa}{y} + \frac{1 - 4\nu^2}{4y^2} \right) = 0. \tag{2.1}$$

Suppose that h is a real analytic function on $\mathcal{H}$ of weight 2 that is harmonic in the sense that

$$\Delta_2 h = y^2(\partial_x^2 + \partial_y^2)h - i2y(\partial_x + i\partial_y))h = 0. \tag{2.2}$$

Then h has a (unique) Fourier expansion in the cusp at ∞ of the form

$$h(z) = \sum_n b(n)\mathcal{M}_n(y)e(nx) + \sum_n a(n)\,\mathcal{W}_n(y)e(nx). \tag{2.3}$$

where

$$\mathcal{W}_n(y) = \begin{cases} (4\pi|n|y)^{-1}W_{-1,1/2}(4\pi|n|y) & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ e^{-2\pi ny} & \text{if } n > 0, \end{cases} \tag{2.4}$$

$$\mathcal{M}_n(y) = \begin{cases} |n|e^{-2\pi ny} & \text{if } n < 0, \\ (4\pi y)^{-1} & \text{if } n = 0, \\ (4\pi y)^{-1}\mathfrak{M}(4\pi ny) - (1-\gamma)ne^{-2\pi ny} & \text{if } n > 0. \end{cases} \tag{2.5}$$

Here $W_{-1,1/2}$ is defined in (2.7), $\mathfrak{M}$ is defined in Lemma 1 and γ is the Euler constant.

For h to be a harmonic weak Maass form we ask that in its Fourier expansion (2.3) only finitely many $b(n) \neq 0$. The above normalization of the special function $\mathcal{M}_n(y)$ is chosen so that the Poincaré series constructed in the next section have only one exponentially growing term in their Fourier expansions and the formulas for their holomorphic coefficients can be given uniformly in terms of a Rademacher type sum.

For fixed μ, ν with $\mathrm{Re}(\nu \pm \mu + 1/2) > 0$ the Whittaker functions may be defined for $y > 0$ by [14, pp. 311, 313]

$$M_{\mu,\nu}(y) = y^{\nu+\frac{1}{2}}e^{\frac{y}{2}}\,\tfrac{\Gamma(1+2\nu)}{\Gamma(\nu+\mu+\frac{1}{2})\Gamma(\nu-\mu+\frac{1}{2})}\int_0^1 t^{\nu+\mu-\frac{1}{2}}(1-t)^{\nu-\mu-\frac{1}{2}}e^{-yt}\,dt \quad \text{and} \tag{2.6}$$

$$W_{\mu,\nu}(y) = y^{\nu+\frac{1}{2}}e^{\frac{y}{2}}\,\tfrac{1}{\Gamma(\nu-\mu+\frac{1}{2})}\int_1^\infty t^{\nu+\mu-\frac{1}{2}}(t-1)^{\nu-\mu-\frac{1}{2}}e^{-yt}\,dt. \tag{2.7}$$

Their asymptotic behavior as $y \to \infty$ for fixed μ, ν is easily found from (2.6) and (2.7) by changing variable $t \mapsto t/y$: As $y \to \infty$

$$M_{\mu,\nu}(y) \sim \tfrac{\Gamma(1+2\nu)}{\Gamma(\nu-\mu+\frac{1}{2})}\,y^{-\mu}e^{y/2} \quad \text{and} \quad W_{\mu,\nu}(y) \sim y^{\mu}e^{-y/2}. \tag{2.8}$$

If $h(x+iy) = y^{-k/2}e(nx)(aM_{(\mathrm{sign}\,n)k/2,s-1/2}(4\pi|n|y) + bW_{(\mathrm{sign}\,n)k/2,s-1/2}(4\pi|n|y))$ then

$$\Delta_k h = (s-k/2)(1-k/2-s)h. \tag{2.9}$$

We will need the following facts about Whittaker functions. When $\mathrm{sign}(n) = -1$, at the special value of the parameter $s = k/2$,

$$M_{-k/2,k/2-1/2}(y) = y^{k/2}e^{y/2}. \tag{2.10}$$

$$W_{-k/2,k/2-1/2}(y) = y^{k/2}e^{y/2}\Gamma(1-k,y) \tag{2.11}$$

where $\Gamma(s,y) = \int_y^\infty e^{-t}t^{s-1}dt$ is the incomplete Gamma function. Moreover for $n < 0$,

$$\xi_k(y^{-k/2}W_{-k/2,k/2-1/2}(4\pi|n|y)e(nx)) = -(4\pi|n|)^{1-k/2}q^{|n|}$$

On the other hand when $\mathrm{sign}(n) = 1$, at $s = k/2$, the two Whittaker functions coalesce;

$$M_{k/2,(k-1)/2}(y) = W_{k/2,(k-1)/2}(y) = y^{k/2}e^{-y/2}. \tag{2.12}$$

In this special case we use the classical method of taking derivatives in s (cf. [5])) to construct a growing solution of Whittaker's equation. We have

Proposition 1. [5] *For $m, k > 0$ positive integers, let*

$$\mathfrak{M}(y) = \partial_s|_{s=k/2}\left(M_{k/2,s-1/2} - W_{k/2,s-1/2}\right)(y), \tag{2.13}$$

and

$$\psi_{m,k}(z) = (4\pi y)^{-k/2}\mathfrak{M}(4\pi m y)e(mx) \tag{2.14}$$

Then

$$D_{k/2,k/2-1/2}\mathfrak{M}(y) = 0, \tag{2.15}$$

$$\Delta_k(\psi_{m,k}(z)) = 0, \tag{2.16}$$

and

$$\xi_k(\psi_{m,k}(z)) = m^{1-k/2}(4\pi)^{1-k}(k-1)\Gamma(k/2)e(-mz). \tag{2.17}$$

Moreover, as $y \to \infty$

$$\mathfrak{M}(y) \sim (k-1)\Gamma(k/2)y^{-k/2}e^{y/2}. \tag{2.18}$$

Proof. If $D_{\kappa,\nu}$ is the Whittaker differential operator as above then

$$D_{\kappa,s-1/2}\partial_s M_{\kappa,s-1/2}(y) = (2s-1)y^{-2}M_{\kappa,s-1/2}(y). \tag{2.19}$$

Similarly

$$D_{\kappa,s-1/2}\partial_s W_{\kappa,s-1/2}(y) = (2s-1)y^{-2}W_{\kappa,s-1/2}(y). \tag{2.20}$$

When $\kappa = k/2$ and $s = k/2$, since $W_{k/2,s-1/2} = M_{k/2,s-1/2}$, it follows by seperation of variables that $\Delta_k(\psi_{m,k}(z)) = 0$. This in turn implies that $\xi_k\psi_{m(,k}z)$ is holomorphic, since $\Delta_k = -\xi_{2-k}\circ\xi_k$. But a simple computation shows that $\xi_k(\psi_{m,k}(z)) = f(y)e(-mx)$ for some function f, and the only holomorphic function with this property is a multiple of $e(-mz)$. Hence $\xi_k(\psi_{m,k}(z)) = ce(-mx)$ for some constant c. The exact value of the constant c follows from the asymptotic behaviour of $\mathfrak{M}(y)$.

To see the asymptotic behaviour of $\mathfrak{M}(y)$ as $y \to \infty$ we use the integral representations of Whittaker functions valid when $\mathrm{Re}\, s > k/2$. Writing $M_{k/2,s-1/2}(y) = e^{y/2}g_s(y)h_s(y)$ with

$$g_s(y) = \frac{y^s\Gamma(2s)}{\Gamma(s+k/2)\Gamma(1+s-k/2)} \tag{2.21}$$

and

$$h_s(y) = (s-k/2)\int_0^1 t^{k/2+s-1}(1-t)^{s-k/2-1}e^{-yt}dt \tag{2.22}$$

gives

$$\partial_s M_{k/2,s-1/2(y)} = e^{y/2}g_s(y)\partial_s h_s(y) + e^{y/2}\partial_s g_s(y)h_s(y) \tag{2.23}$$

Integration by parts in $h_s(y)$ leads to

$$h_s(y) = \int_0^1 (1-t)^{s-k/2}\,\partial_t(t^{k/2+s-1}e^{-yt})dt. \tag{2.24}$$

As a byproduct this immediately gives $h_{k/2}(y) = e^{-y}$ proving (2.12).

We also have

$$\partial_s h_s(y) = \int_0^1 (1-t)^{s-k/2} \ln(1-t) \partial_t (t^{s+k/2-1} e^{-yt}) dt + \int_0^1 (1-t)^{s-k/2} \partial_t (\partial_s (t^{s+k/2-1} e^{-yt})) dt.$$

At $s = k/2$ this leads to

$$\partial_s|_{s=k/2} h_s(y) = \int_0^1 \ln(1-t) \partial_t (t^{k-1} e^{-yt}) dt$$

By Watson's lemma from asymptotic analysis (see [17] or [13], p. 467)

$$-\int_0^1 e^{-yt} (yt - k + 1) t^{k-2} \ln(1-t) dt \sim (k-1) y^{-k} \Gamma(k).$$

The function $W_{k/2,s-1/2}$ is treated analogously leading to

$$\partial_s|_{s=k/2} \int_1^\infty t^{s+\frac{k}{2}-1} (t-1)^{s-\frac{k}{2}-1} e^{-yt} \, dt = \int_1^\infty e^{-yt} (yt - k + 1) t^{k-2} \ln(t-1) dt,$$

which is easily seen to be $O(e^{-y})$.

To summarize we have

$$\mathfrak{M}(y) \sim (k-1)\Gamma(k/2) y^{-k/2} e^{y/2}.$$

The asymptotic expansion of $\xi_k(\psi_{m,k}(z))$ follows along similar lines proving (2.17).

$\square$

3. Poincaré series

In this section we compute the Fourier expansion of a Poincaré series and establish some needed analytic properties of the Fourier coefficients. Let $\phi(y) : \mathbb{R}_{>0} \to \mathbb{C}$ be smooth and such that for some $\alpha > 0$, $\phi(y) = O(y^\alpha)$. For $0 \neq m \in \mathbb{Z}$ let

$$\phi_m(z) = |m|^{k/2} \phi(|m|y) e(mx)$$

and define the Poincaré series

$$P_m(z, \phi) = P_m(z, \phi, k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\phi_m|_k \gamma)(z).$$

Then $P_m(z, \phi)$ converges for $\alpha > 1 - k/2$ and has weight k.

We will work with a special ϕ depending on a parameter s as well as its derivative in s. Let

$$\phi_m(z, s) = (4\pi y)^{-k/2} M_{\mathrm{sign}(m)k/2, s-1/2}(4\pi|m|y) e(mx)$$

and

$$P_m(z, s) = P(z, \phi_m(z, s)).$$

$P_m(z, s)$ converges for $\mathrm{Re}\, s > 1$ and is harmonic at $s = k/2, 1 - k/2$. For $k > 2$ this gives immediately harmonic forms. But these Poincaré series only lead to either holomorphic cusp forms or weakly holomorphic modular forms since at the special parameters

$$M_{\pm k/2,(k-1)/2}(y) = y^{k/2} \frac{e^{\mp y/2}}{\Gamma(k)}.$$

For $m > 0$, $P_m(z, k/2)$ reduces to the classical exponential Poincaré series which is a cusp form and for $m < 0$ one can only get growing terms of the form $e^{2\pi i m z}$ and hence it leads to a weakly holomorphic modular form.

3.1. **Fourier coefficients.** We are interested in the case of $k = 2$, when convergence of the Poincaré series is not known at the parameter which makes it harmonic. We will continue it analytically to the point we are interested in by its Fourier expansion. The Fourier coefficients of Poincaré series can be expressed explicitly in terms of sums Kloosterman sums for any weight k. The following Proposition is standard and can be found for example in [9], [3].

Proposition 2. *With the notation as above*

$$P_m(z,s) = \phi_m(z,s) + g_{m,0}(s)L_{m,0}(s)(-4\pi)^{-k/2}y^{-k/2+1-s} \sum_{n\neq 0} g_{m,n}(s,y)L_{m,n}(s)(-4\pi y)^{-k/2}W_{\mathrm{sign}(n)k/2,s-1/2}(4\pi|n|y)e(nx) \tag{3.1}$$

where

$$g_{m,n}(s) = \Gamma(2s)\begin{cases} \dfrac{2\pi\sqrt{|m/n|}}{\Gamma(s+\mathrm{sign}(n)k/2)} & \text{if } n\neq 0 \\[2ex] \dfrac{4\pi^{1+s}|m|^s}{(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)}, & \text{if } n=0 \end{cases} \tag{3.2}$$

and

$$L_{m,n}(s) = \begin{cases} \sum_c \dfrac{K(m,n,c)}{c} J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right), & \text{if } \mathrm{sign}(nm)>0 \\ \sum_c \dfrac{K(m,0,c)}{c^{2s}}, & \text{if } n=0 \\ \sum_c \dfrac{K(m,n,c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right), & \text{if } \mathrm{sign}(nm)<0 \end{cases} \tag{3.3}$$

and where $K(m,n,c)$ is the Kloosterman sum

$$K(m,n,c) = \sum_{a(c)} e\left(\tfrac{m\bar{a}+na}{c}\right).$$

In view of $W_{k/2,k/2-1/2}(y) = y^{k/2}e^{-y/2}$ for $s = k/2$ ($k \geq 2$) we have

$$g_{m,n}(k/2)(-4\pi y)^{-k/2}W_{\mathrm{sign}(n)k/2,k/2-1/2}(4\pi|n|y)e(nx) = \begin{cases} (-1)^{k/2}(2\pi)\sqrt{|m/n|}\,n^{k/2}q^n & \text{if } n>0 \\ 0 & \text{if } n<0 \end{cases} \tag{3.4}$$

Hence if $m > 0$, and $k > 2$, at $s = k/2$, (3.1) leads to the the Fourier expansion of the classical cuspidal Poincaré series and if $m < 0$ this construction fails to produce a weak harmonic form that is not weakly holomorphic since

$$M_{-k/2,(k-1)/2}(y) = y^{k/2}\frac{e^{y/2}}{\Gamma(k)}. \tag{3.5}$$

3.2. **Analytic continuation.** We will show that $L_{m,n}(s)$ can be analytically continued to $\operatorname{Re} s > 3/4$ with only possible pole for $n = 0$ at $s = 1$. We will also get estimates for $L_{m,n}$ which will ensure the convergence of the Fourier series for $\operatorname{Re} s > 3/4 + \epsilon$ for any $\epsilon > 0$ and this gives the analytic continuation of $P_m(z, s)$ to $s = 1$. The case $\operatorname{sign}(nm) > 0$ is worked out in [10], Section 16.5.

$$L_{m,n}(s) \ll (2\sigma - 1)^{-2} e^{\pi|s|/2} \left(\frac{\sqrt{mn}}{|s|} \right)^{\sigma}$$

The case $\operatorname{sign}(nm) < 0$ goes along the same lines but will lead to an estimate which is qualitatively different due to the growth of I_s at ∞. In this case

$$L_{m,n}(s) = \sum_{c \le \sqrt{|mn|]}} \frac{K(m,n,c)}{c} I_{2s-1} \left(\frac{4\pi\sqrt{|mn|}}{c} \right) + \sum_{c > \sqrt{|mn|}} \frac{K(m,n,c)}{c} I_{2s-1} \left(\frac{4\pi\sqrt{|mn|}}{c} \right)$$

Using

$$I_\nu(y) = \frac{y^\nu}{2^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^\pi e^{y \cos\theta} \sin^{2\nu} \theta d\theta$$

valid when $\operatorname{Re} \nu + \frac{1}{2} > 0$, we have that for $\frac{1}{4} < \operatorname{Re} s < 2$

$$I_{2s-1} \left(\frac{4\pi\sqrt{|mn|}}{c} \right) \ll \begin{cases} \dfrac{|mn/c^2|^{\operatorname{Re} s - \frac{1}{2}}}{|\Gamma(2s - \frac{1}{2})|} & \text{when } c > \sqrt{|mn|} \\ e^{4\pi\sqrt{|mn|}/c} \dfrac{|mn/c^2|^{\operatorname{Re} s - \frac{1}{2}}}{|\Gamma(2s - \frac{1}{2})|} & \text{when } c \le \sqrt{|mn|} \end{cases}$$

The first sum estimated trivially for $\sigma = \operatorname{Re} s > 3/4$

$$\sum_{c \le \sqrt{|mn|]}} \frac{K(m,n,c)}{c} I_{2s-1} \left(\frac{4\pi\sqrt{|mn|}}{c} \right) \ll e^{4\pi\sqrt{|mn|}} \frac{|mn|^{\operatorname{Re} s}}{|\Gamma(2s - \frac{1}{2})|}$$

For the second sum we use eqn. (16.50) of [10]

$$\sum_c \frac{|K(m,n;c)|}{c^{2\sigma}} \ll (\sigma - 3/4)^{-2}$$

to get

$$\left| \sum_{c > \sqrt{|mn|}} \frac{K(m,n,c)}{c} I_{2s-1} \left(\frac{4\pi\sqrt{|mn|}}{c} \right) \right| \ll \frac{|mn|^{\operatorname{Re} s - \frac{1}{2}}}{|\Gamma(2s - \frac{1}{2})|} (\sigma - 3/4)^{-2}$$

For $n = 0$, we note that $g_{m0}(s) L_{m0}(s) = .\dfrac{4\pi|\pi m|^{s-k/2}\Gamma(2s)}{(-2)^k(2s-1)\Gamma(s+k/2)\Gamma(s-k/2)} \dfrac{\sigma_{2s-1}(m)}{m^{2s-1}\zeta(2s)}$ is analytic for $\sigma > 3/4$. This gives the analytic continuation of the Fourier coefficients.

To get the continuation of $P(z, s)$ to $s = 1$ we still need to prove the convergence of the Fourier series. But this follows from the exponential decay of the Whittaker function $W_{\mu,\nu}(y)$, namely that as $y \to \infty$ $W_{\mu,\nu}(y) \sim y^\mu e^{-y/2}$.

3.3. **Weight 2.** Let $k = 2$ and set $P_m(z) = P_m(z, 1)$. Then we have

Proposition 3. *If $m > 0$ then $P_m(z) = 0$. If $m < 0$ then*

$$P_m(z) = |m|q^m - \sum_{n>0} nc_{|m|}(n)q^n = f'_{|m|} \tag{3.6}$$

Proof. For $m > 0$, due to the Gamma factors in the denominators in (3.2), $a(n, y) = 0$ for $n \leq 0$ and (3.2) gives the Fourier expansion of the holomorphic Poincaré series. Since for $\mathrm{SL}(2, Z)$ there are no cusp forms of weight 2, we have in fact $P_m(z, 1) \equiv 0$ for $m > 0$. On the other hand for $m < 0$, $\phi_m(z, 1) = |m|q^m$, $a(n, y) = 0$ for $n < 0$ and using (1.2) we see that (3.2) in this case match exactly the formulas of Rademacher for $f'_{|m|}(z)$. Hence for $m < 0$, $P_m(z, 1) = f'_{|m|}(z)$. □

Remark. Alternatively we may prove the Petersson-Rademacher formulae using the fact that $P_m - f'_{|m|}$ is a holomorphic cusp form of weight 2.

For $m = 0$ we set

$$P_0(z) = E_2^*(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n - \frac{3}{\pi y}. \tag{3.7}$$

4. Non-holomorphic harmonic weak Maass forms

4.1. **Derivatives of Poincaré series.** As we have seen in the last section the above construction of Poincaré series $P_m(z)$ only produces holomorphic cusp forms for $m > 0$ or weakly holomorphic forms in case $m < 0$. To write down the basis for harmonic weak Maass forms we need to find harmonic forms that are not weakly holomorphic. Such forms will be constructed using derivatives in s of Poincaré series.

For $m > 0$ a growing solution of Whittaker's equation can be constructed as in Section 2 by taking the derivatives in s of the Whittaker functions.

Recall that if $D_{\kappa,\nu} = \frac{d^2}{dy^2} + (-\frac{1}{4} + \frac{\kappa}{y} + \frac{1-4\nu^2}{4y^2})$ is the Whittaker differential operator then

$$D_{k/2,s-1/2}\partial_s M_{k/2,s-1/2}(y) = (2s-1)y^{-2}M_{\kappa,s-1/2}. \tag{4.1}$$

$$D_{k/2,s-1/2}\partial_s W_{k/2,s-1/2}(y) = (2s-1)y^{-2}W_{k/2,s-1/2}. \tag{4.2}$$

Therefore we have that for $\psi_m(z, s) = (4\pi y)^{-k/2}\partial_s \left(M_{k/2,s-1/2} - W_{k/2,s-1/2}\right)(4\pi my)e(mx)$ the Poincaré series $P_m(z, \psi(s, .))$ is harmonic at $s = k/2$.

To prove that the Poincaré series is meaningful note that for $k > 2$ we may simply differentiate the Poincaré series built from $M_{\kappa,s-1/2}(y)$ or $W_{k/2,s-1/2}(y)$ since $s = k/2$ is in $\operatorname{Re} s > 1$, which is a region of local uniform convergence of theses series. For $k = 2$ this no longer holds and we modify the argument as follows.

4.2. **Weight 2.** Recall that for $m > 0$ the Poincaré series $P_m(z, s)$ was built from

$$\phi_m(z, s) = (4\pi y)^{-k/2}M_{k/2,s-1/2}(4\pi my)e(mx).$$

Since $P_m(z, s)$ has an analytic continutaion to $\operatorname{Re} s > 3/4$ by (2.9) we have

$$\Delta_k(\partial_s P_m(z, s)) = (2s-1)(P_m(z, s))$$

Since $P_m(z, 1) \equiv 0$ in the case of $k = 2$ we have that

$$Q_m(z) = \partial_s|_{s=1}P_m(z, s)$$

is already harmonic and the Fourier expansion reveals only one growing term, so it is a weak harmonic Maass form.

4.3. **Fourier coefficients of Q_m.** To find the Fourier expansion of $Q_m(z)$ we differentiate the coefficients given in Proposition 2 to get

Proposition 4. *Let $Q_m(z) = \partial_s|_{s=1} P_m(z,s)$ then*

$$Q_m(z) = \mathcal{M}_m(y)e(mx) - \frac{6}{\pi y}\sigma_1(m) - \sum_{n<0} |n| c_m(|n|)\mathcal{W}_n(y)e(nx) - \sum_{n>0} \mathcal{L}_{mn} q^n \tag{4.3}$$

where

$$\mathcal{L}_{mn} = 2\pi\sqrt{nm}\sum_c \frac{K(m,n,c)}{c} F(\frac{4\pi\sqrt{|mn|}}{c}) \tag{4.4}$$

with $F(u) = \partial_s|_{s=1} J_{2s-1}(u)$ and γ the Euler's constant.

Theorem 1 follows from the formula $\partial_s|_{s=1} J_s(x) = \frac{\pi}{2} Y_1(x) + \frac{1}{x} J_1(x)$.

Proof.

$$\begin{aligned} Q_m(z,s) = \partial_s P_m(z,s) &= \partial_s \phi_m(z,s) + (-4\pi)^{-1}\partial_s (g_{m,0}(s)L_{m,0}(s)y^{-s}) \\ &+ \sum_{n\neq 0} (-4\pi y)^{-1} \partial_s \left[g_{m,n}(s)L_{m,n}(s)\right] W_{\mathrm{sign}(n),s-1/2}(4\pi|n|y)e(nx) \\ &+ \sum_{n\neq 0} (-4\pi y)^{-1} g_{m,n}(s)L_{m,n}(s)\partial_s \left[W_{\mathrm{sign}(n),s-1/2}(4\pi|n|y)\right] e(nx) \end{aligned} \tag{4.5}$$

where $g_{m,n}$ and $L_{m,n}$ are defined in (3.2) and (3.3). Since there are no cusp forms of weight 2 for $SL(2,\mathbb{Z})$ it follows that $g_{m,n}(1)L_{m,n}(1) = 0$ for all $n \neq m$, and that $g_{m,m}(1)L_{m,m}(1) = 1$.

First note that a calculation using $g_{m0}(s)L_{m0}(s) = \dfrac{4\pi|\pi m|^s\Gamma(2s)}{(2s-1)\Gamma(s+1)\Gamma(s-1)} \dfrac{\sigma_{2s-1}(m)}{m^{2s-1}\zeta(2s)}$, gives the constant coefficient as $-\dfrac{6}{\pi y}\sigma_1(m)$.

To compute the derivatives we will treat the cases $n > 0$ and $n < 0$ separately.

When $n < 0$, $g_{m,n}(1) = 0$, and hence the contribution of the second sum in (4.5) to $Q_m(z)$ is zero. Since for $n < 0$, and $\partial_s|_{s=1} g_{m,n} = 2\pi\sqrt{|m/n|}$, the first sum in (4.5) contrubutes

$$-\sum_{n<0} 2\pi L_{mn}(1)\sqrt{|mn|}(4\pi|n|y)^{-1}W_{-1,1/2}(4\pi|n|y)e(nx) = -\sum_{n<0}|n|c_m(|n|)\mathcal{W}_n(y)e(nx)$$

On the other hand, if $n > 0$, $W_{1,1/2}(y) = ye^{-y/2}U(0,2,y) = ye^{-y/2}$ and $g_{m,n}(1)L_{m,n}(1) = \delta_{m,n}$. Therefore, for $m,n > 0$, second sum's contribution is $-(4\pi y)^{-1}\partial_s|_{s=1}W_{1,s-1/2}(4\pi m y)$ where as the first sum contributes

$$-\sum_{n>0}\partial_s|_{s=1}\left[g_{m,n}(s)L_{m,n}(s)\right]nq^n = -\sum\left[g_{mn}(1)\partial_s|_{s=1}L_{m,n}(s) + (\partial_s|_{s=1}g_{mn}(s))L_{mn}(1))\right]nq^n$$

Let $F(u) = \partial_s|_{s=1}J_{2s-1}(u)$, using $g_{m,n}(1) = 2\pi\sqrt{|m/n|}$ and $\partial_s|_{s=1}g_{m,n} = (1-\gamma)2\pi\sqrt{|m/n|}$ and $L_{m,n}(1) = \delta_{mn}/2\pi$ we see that for $n > 0$ the first sum (4.5) at $s = 1$ gives

$$-m(1-\gamma)q^m - \sum_{n>0}\mathcal{L}_{mn}q^n \tag{4.6}$$

where

$$\mathcal{L}_{mn} = 2\pi\sqrt{mn}\sum_c \frac{K(m,n,c)}{c}F(\frac{4\pi\sqrt{|mn|}}{c}). \tag{4.7}$$

Note that $-(4\pi y)^{-1}\partial_s|_{s=1}W_{1,s-1/2})(4\pi my)e(mx)$ together with the term $-m(1-\gamma)q^m$ is exactly what is needed to combine with $\partial_s|_{s=1}\phi_m(z,s)$ to give the harmonic growing term $\mathcal{M}_m(y)e(mx)$ in (4.3).

□

We can now give a basis for $H_2^!$

Proposition 5. *Let*

$$h_m(z) = \begin{cases} f'_{|m|}(z) & \text{if } m<0, \\ E_2^*(z) & \text{if } m=0, \\ Q_m(z) & \text{if } m>0, \end{cases} \tag{4.8}$$

Then for each $m\in\mathbb{Z}$, $h_m(z)$ has a Fourier expansion of the form

$$h_m(z) = \mathcal{M}_m(y)e(mx) + \sum_{n\in\mathbb{Z}} a_m(n)\mathcal{W}_n(y)e(nx). \tag{4.9}$$

The coefficients satisfy the symmetry relation $a_m(n)=a_n(m)$ for all $m,n\in Z$.

Moreover $\{h_m\}_{(m\in\mathbb{Z})}$ form a basis for $H_2^!$ and if $m>0$, $\quad \xi_2(h_m)(z) = \dfrac{1}{4\pi}f_m(z).$

Proof. It follows from the Fourier expansions (3.6), (4.3), (3.7) of $P_m(z)$, $Q_m(z)$ and $E_2^*(z)$ that $h_m(z)$ has the expansion of the desired form (4.9).

The symmetry relation $a_m(n)=a_n(m)$ follows from the symmetry relations $|n|c_{|m|}(|n|) = |m|c_{|n|}(|m|)$ and $\mathcal{L}_{mn}=\mathcal{L}_{nm}$.

If $h(z)\in\mathcal{H}_2^!$ is a weak Harmonic form, then it has a Fourier expansion of the form (2.3) with only finitely many coefficients $b(n)$.Hence $h(z)-\sum_{n\neq 0}b(n)h_m(z)$ is a Harmonic weak Maass form with no growing term in its Fourier expansion, hence is a constant multiple of E_2^*.

Now the fact that for $n<0$, $\xi_2(\mathcal{W}_n(y)e(nx)) = -(4\pi|n|)^{-1}q^{|n|}$ together with (1.2), (2.17), and (4.3) proves that $\xi_2(h_m)(z) = \dfrac{1}{4\pi}\left[j_m(z)+24\sigma_1(m)\right] = \dfrac{1}{4\pi}f_m(z)$.

□

The above proposition shows that the non-holomorphic coefficients of the basis functions h_m for $m>0$ are the coefficients $|n|c_m(n)$ of f'_m. On the other hand the holomorphic coefficients of h_m are complicated expressions involving sums of Kloosterman sums multiplied by the derivatives of the Bessel function in the index, $F(u)$. $F(u)$ can be expressed in various forms, one of which is the formula $F(u)=\pi Y_1(u)+\frac{2}{u}J_0(u)$ given in Theorem 1. However we will give an interpretation of these holomorphic Fourier coefficients in the next section.

5. Inner products

In this section we will show that holomorphic Fourier coefficients of the harmonic forms of weight 2 constructed in the previous section can be written in terms of the regularized Petersson inner products of f_n and f_m.

For this purpose we need the following lemma which follows from a standard application of Stokes theorem see [1].

Lemma 1. *Suppose $k\in\mathbb{Z}$ and that g is holomorphic on $\mathcal{H}$ of weight k for Γ. Suppose that h is a smooth function of weight $2-k$. Then, for $Y\geq 2$ we have*

$$\int_{\mathcal{F}_Y} g(z)\overline{\xi_{2-k}h(z)}y^k\,\tfrac{dxdy}{y^2} = \int_{-1/2+iY}^{1/2+iY} g(z)h(z)dz.$$

Theorem 4. *Suppose that m, n are distinct positive integers. Then*

$$\langle f_n, f_m\rangle = -4\pi\mathcal{L}_{mn}.$$

Proof. We have that for $m > 0$, h_m satisfies $\xi_2 h_m = \dfrac{1}{4\pi} f_m$ and its Fourier expansion has the form

$$h_m(z) = Q_m(z) = \mathcal{M}_m(y)e(mx) - \frac{6}{\pi y}\sigma_1(m) - \sum_{n<0}|n|c_m(|n|)\mathcal{W}_n(y)e(nx) - \sum_{n>0}\mathcal{L}_{mn}q^n$$

where $\mathcal{M}_m(y) \sim e^{2\pi m y}$ as $y \to \infty$ is the lone exponentially growing term.

We have the following bounds for the Fourier coefficients of h_m. First by classical estimates [15] we have that $c_m(n) = O(e^{C\sqrt{n}})$ and by [4] (see Lemma 3.4), we also have $\mathcal{L}_{mn} = O(e^{C\sqrt{n}})$. It follows using Lemma 1 that

$$\int_{\mathcal{F}_Y} f_n(z)\overline{\xi_2 h_m(z)}\, dxdy = \int_{-1/2+iY}^{1/2+iY} f_n(z)h_m(z)dz$$

Now $f_n(z) = q^{-n} + \sum_{k\ge 0} c_n(k)q^k$ and

$$h_m(z) = \mathcal{M}_m(y)e(mx) - \frac{6}{\pi y}\sigma_1(m) - \sum_{l<0}|l|c_m(|l|)\mathcal{W}_l(y)e(lx) - \sum_{l>0}\mathcal{L}_{ml}q^l$$

By the above estimates these sums can be multiplied and integrated term by term leading to

$$\lim_{Y\to\infty}\int_{\mathcal{F}_Y} f_n(z)\overline{\xi_2 h_m(z)}\, dxdy = -\mathcal{L}_{mn} - i\lim_{Y\to\infty}\sum_{k>0} kc_n(k)c_m(k)\mathcal{W}_k(Y)e^{-2\pi kY} = -\mathcal{L}_{mn}.$$

□

Since $\xi_2(h_m)(z) = \frac{1}{4\pi}f_m(z)$ this, together with (4.7), proves the formula in Theorem 1. It also gives a complete description of the Fourier coefficients of weak harmonic Maass forms in weight 2 and/or the values of the inner products $\langle f_n, f_m\rangle$, depending on one's point of view. Theorem 4 together with Proposition 5 also finishes the proof of Theorem 2.

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